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ANALYSIS OF STUDENTS' UNDERSTANDING OF ALGEBRA CONCEPTS BASED ON APOS THEORY: A STUDY IN LINEAR ALGEBRA COURSES

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ABSTRACT
Understanding algebraic concepts is an important prerequisite for advanced mathematics learning because it involves a transition from symbolic manipulation to structural thinking and proof. This study analyzes students' understanding of algebraic concepts in a Linear Algebra course using the APOS (Action, Process, Object, Schema) theoretical framework. The method is descriptive qualitative, with systematic application of APOS indicators in the domain of Linear Algebra. Data were collected through conceptual tests and analysis of proof and problem-solving tasks. Data analysis involved reduction and categorization based on APOS development. The results showed three patterns of understanding: (1) students in the low-ability category failed to meet the criteria for the Action, Process, Object, and Schema stages; (2) the medium category showed partial comprehension—reaching the Process stage and parts of the Object stage on routine problems but failing at the Action stage for abstract constructive problems; and (3) the high category generally reached the Schema stage on constructive problems but showed failure at the Action stage on problems requiring proofs of basic axioms, or vice versa. The findings suggest emphasizing development of the Action stage and the process–object linkages to strengthen transfer to the Schema. Practical recommendations include gradual constructive tasks, mediated proof-of-concept learning, and APOS-indicator-based formative assessment.

INTRODUCTION

Mathematics plays a central role in the development of science and technology and in shaping logical thinking and problem-solving skills. As a universal language for modeling natural, social, and technological phenomena, mathematics not only supports advances in other scientific disciplines but also strengthens students' analytical and critical abilities. (Dewimarni, 2017). Therefore, mastery of mathematical concepts is an important prerequisite for academic and professional success across many fields.

Linear Algebra is an essential foundational course in college, especially for students in mathematics, engineering, and science programs, because it underpins advanced subjects such as numerical analysis, linear transformations, and applications in computing and machine learning. The course introduces abstract concepts such as vectors, matrices, vector spaces, and linear transformations, that require deep conceptual understanding beyond procedural skills (Zuraidah et al., 2026). According to Mufidah, *et.al.* (2019), Algebra is a crucial branch of mathematics because it extends and generalizes arithmetic, which itself is a foundation of mathematics. Mastery of algebraic concepts is an important competency in mathematics learning, characterized by the ability to relate concepts and to apply algorithms flexibly, accurately, efficiently, and appropriately in problem solving. Therefore, success in this course is important for developing the mathematical competence of prospective mathematics teachers.

Despite the importance of the Linear Algebra course, observations show that students often have difficulty understanding its abstract concepts. Common problems include interpreting the concept of a vector space, relating geometric and algebraic representations, and using symbols and operations in the context of linear transformations. Factors that contribute to students' weak understanding of algebraic concepts include low mathematical modeling skills, difficulties with conceptual thinking, and limited learning activities that facilitate the construction of meaning.

Various studies show that students at different levels continue to have difficulty understanding algebraic and other mathematical concepts. Common errors include incorrect symbolic manipulation, limited conceptual understanding, and difficulty translating between multiple representations. These barriers also appear in specific topics such as algebra and calculus, where students struggle to interpret abstract concepts, model real-world problems, and articulate their reasoning. Although some students master basic principles, many still have trouble with areas such as geometry and complex problem solving; therefore, evidence-based interventions are needed to bridge both conceptual and procedural gaps (Kania et al., 2025). Vitantri and Umah (2020) stated that students struggle to understand concepts because they cannot relate new material to prior knowledge, so they often forget earlier topics when new ones are introduced. In addition, variation in student ability requires instructors to design appropriate learning methods so that students with lower abilities can also understand the material.

The results of Tashtoush, *et.al* (2023) show that, overall, students' conceptual understanding and problem-solving skills regarding systems of linear equations (SLE) are relatively low. In terms of their ability to use multiple representations (algebraic and graphical), mastery falls into the medium category, indicating qualitative differences in understanding between representations. Most students tend to feel more comfortable with algebraic representations, likely due to prior learning habits and teaching practices, while their ability to use graphical representations (lines, curves, intersection points) is relatively weak. Common misconceptions in algebraic representations include errors in row operations, distribution, interpretation of parentheses, and incorrect algebraic generalizations. On the graphical side, difficulties include locating intersection points, determining whether lines intersect or are parallel, and matching systems of equations to their graphical illustrations. These findings point to the need to strengthen instructional approaches that integrate dual-representation exercises and emphasize conceptual understanding rather than mere procedural mastery.

Research by Hanifah and Masruroh (2022) found that students often have difficulty understanding and solving problems in Linear Algebra, particularly in the topic of matrix determinants. One main cause is low precision in carrying out solution steps, so even when students understand basic procedures, small errors in calculation or operation application frequently occur and reduce answer accuracy. Other factors that lower learning achievement include low interest in learning, limited cognitive ability, and difficulty applying Linear Algebra concepts to various situations. This situation is exacerbated by a learning approach that tends to be mechanistic and emphasizes memorization over deep conceptual understanding (Hanifah, 2022).

APOS (Action, Process, Object, Schema) theory is a constructivist framework used to address difficulties in learning mathematics (Mulyono, 2011; Rahmani et al., 2024). According to Ed Dubinsky (2000), APOS assumes that mathematical knowledge is formed through social interaction and the individual's mental construction of mathematical ideas (Muttaqin et al., 2019). The theory proposes that the development of mathematical concepts proceeds through four sequential mental structures - action, process, object, and schema - each building on the previous one (Kania et al., 2025). APOS is described as follows (Muttaqin et al., 2019): (1) Action: a transformation of externally presented objects carried out explicitly by following memory or step-by-step instructions. At this stage, the student follows a visible procedure to perform an operation, (2) Process: the transformation moves into the internal realm; students imagine and mentally control the changes so that these actions become part of their thinking, (3) Object: the student begins to view the transformation (the process) as a single entity that can be manipulated and treated as a mathematical object, (4) Schema: a network that connects actions, processes, and objects (and other schemas when needed) with general principles, forming a framework for solving problems related to the concept.

The four stages are arranged hierarchically and follow a fixed order (Izzatin, 2020; Muttaqin et al., 2019). This aligns with Piaget's view of cognitive development, in which stages are not skipped and there is no regression, although the age at which individuals reach each stage may vary. Dubinsky, *et.al.* (2000) explain that APOS theory is a constructivist approach to learning mathematical concepts. According to this theory, mathematical knowledge is reflected in how a person responds to a mathematical problem, expresses and negotiates that response through social interaction, and then constructs or reconstructs mathematical actions, processes, and objects that are organized into a schema for dealing with similar situations (Mulyono, 2011).

The APOS theory provides a comprehensive model for analyzing the development of an individual's mathematical understanding, describing how initial actions on symbols evolve into internalized processes, then into conceptual objects, and finally into organized schemas. Applying APOS theory to Linear Algebra can help identify students' levels of conceptual development and the mechanisms of transition between stages of understanding. However, the use of this theory in Linear Algebra within Mathematics Education programs remains limited, particularly in linking APOS indicators with specific empirical evidence on topics such as vector spaces and linear transformations. Therefore, this study aims to analyze students' understanding of algebraic concepts in Linear Algebra courses using the APOS theoretical framework by systematically applying APOS indicators to the Linear Algebra domain. The research findings are expected to contribute practically to the design of more effective instruction and theoretically as a basis for further studies of mathematics learning at the university level.

METHODS

Types of Research

This study used a descriptive qualitative approach to provide a detailed and systematic description of students' experiences and cognitive processes when understanding algebraic concepts in the Linear Algebra course. The descriptive approach was selected because the main focus is to map learning phenomena occurring in a natural context, not to test causal relationships between variables.

Research setting

The research was implemented in the even semester of the 2025/2026 academic year in the Mathematics Education Study Program, University of Borneo Tarakan in North Kalimantan.

Research Subject

The research subjects consisted of 20 students in the fourth semester. The selection of subjects was carried out by purposive sampling technique according to the criteria set in the study. From all participants, 2 students were selected representing each ability category: high, medium, and low.

Data collection techniques

The researcher acted as the main instrument and was assisted by a Linear Algebra test sheet. The test consisted of descriptive questions designed to assess students' understanding of algebraic concepts, with analysis based on the APOS theoretical framework. To measure mastery of algebraic concepts in the Linear Algebra course, two problems were given. Details of the Linear Algebra questions are presented in Table 1.

Table 1. Questions for Understanding the Concept of Linear Algebra

Nomor Soal	Permasalahan
1	Prove all vectors that are shaped (a, b, c), with $b = a + c + 1$ is a subspace of $R3$
2	Find a base for the subspace $R4$ and also determine the dimensions of all the vectors that are shaped (a, b, c, d) with $d = a + b, c = a - b$

In the next stage, an in-depth analysis of students' answers was carried out to determine how they understand the concept of algebra in the topic of linear algebra. Student responses were traced using the APOS theoretical framework, and each response was analyzed to determine whether it reflected the action, process, object, or schema stage.

Data Analysis

Qualitative data analysis in this study was conducted through three main stages: data reduction, data presentation, and conclusion drawing. In the data reduction stage, the researcher systematically selected, focused, and organized the data to identify information relevant to the research questions. In the data presentation stage, the findings were described narratively based on the indicators of decision-making ability. In the conclusion drawing stage, the researcher synthesized the data to ensure that the findings were aligned with the research objectives (Miles et al., 2014).

RESULTS AND DISCUSSION

The results of the study indicate differences in the level of understanding of mathematical concepts among the six subjects based on the APOS Theory framework in the Linear Algebra course. The six subjects were classified into three categories of mathematical ability, namely high, medium, and low, with each category represented by two subjects. The low-ability subjects were AY and D, the medium-ability subjects were NHP and RA, and the high-ability subjects were NS and LR. The analysis of each subject's conceptual understanding based on the stages of APOS Theory is presented as follows:

The Low-ability Subject

The results of the analysis of the answers of low-ability subjects in 2 questions to see the students' understanding of Linear Algebra concepts based on APOS theory are described as follows:

1. Understanding Subject Concepts at the Action Stage

The subject AY can be said to have failed in the Action Stage. In question 1 (Subspace): the subject does have the intention to perform a procedural action (entering the number zero). However, the basic rules of algebra are wrong. The action of summing up brackets $(0 + 0 + 1)$ three times in a row is an act of manipulation that is invalid and not based on existing mathematical rules. A qualifying action should be a substitution execution $a = 0, b = 0, c = 0$ exactly on the left and right sections. While in question 2 (Basis): the subject has not executed the main action required by the question at all. The earliest procedural action for the question is to replace the letters c and d with a+b and a-b. Instead, the subject writes

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a series of indexed variables that have nothing to do with the problem. This indicates an irrelevant Action Stage.

Subject D has not met the criteria for Action Stage. In question 1, move the field to $b - a - c = 1$ is actually a legitimate and potentially true form of external Action. However, a fatal failure occurred when the student made a substitution to test the zero vector. From the equation $b - a - c = 1$, If it is substituted with a zero, it should read $0 - 0 - 0$. However, the students instead wrote a series of $0 + 0 + 0$. The subject must at least be able to carry out the counting procedure correctly even if he does not understand the meaning. Failure to perform this basic operation indicates failure at the Action Stage. In question 2, the subjects did not do any action at all, only copied the question.

2. Understanding of Subject Concepts at the Process Stage

The AY subject does not meet the Process Stage criteria. In question 1, the subject does not have the correct operational framework (Process) on how to test the association membership requirements. Instead of subordinating the value $(0,0,0)$ exactly at positions a, b and c in the equation $b = a + c + 1$, Instead, he assembled the sum of brackets haphazardly into $0 = (0 + 0 + 1) + (0 + 0 + 1) + (0 + 0 + 1)$. Since the basic manipulation ability of the Action Stage is mathematically flawed, the subject has not automatically built up a mental construct in the Process Stage. Meanwhile, in question 2, instead of performing the process of substitution and algebraic factoring, the subject writes a series of $= (a_1 + b_1 + c_1 + d_1)+\ldots$ which has nothing to do with the problem. This proves that there is no running problem-solving process, purely relying on a random visual memory of long formulas that have ever been seen, and then simply pasting them onto the answer sheet.

Subject D also does not meet the criteria of the Process Stage. In question 1, the subject starts by moving the field to $b - a - c = 1$. This step is technically correct. But the subject fails to run the flow. Instead of evaluating $0 - 0 - 0$, instead write a series $0 + 0 + 0$ and immediately connect it with the word $= Subruang\ R^{\,3} = 1$". There is no reasoning process that produces contradictions $(0 \neq 1)$. This shows that the subject is only memorizing so it has no Process on how to interpret the results. In question 2, the subjects only copy what is known from the problem. Because it cannot process the initial information into a vector form that is ready to be decomposed.

3. Understanding of Subject Concepts at the Object Level

The AY subject does not meet the object-level criteria in its entirety. In question 1 Object Level it is not just guessing the end result correctly. Objects must be born from a complete understanding of the Process. Here, the subject subordinates the variables very haphazardly: $0 = (0 + 0 + 1) + (0 + 0 + 1) + (0 + 0 + 1)$. Since the Action and Process stages are proven to be flawed (not understanding how substitution works), the final conclusion looks more like a leap of faith or a lucky guess, rather than the result of a careful structural analysis. In question 2, the subjects write the series $= a_1 + b_1 + c_1 + d_1 + \cdots$ and force it to zero. This is clear evidence that the subject does not have an Object Stage of what a parametric basis is. The subject simply regurgitates a visual memory of a series of linear combinations without knowing the context of its use.

Subject D also does not meet the Object Level criteria in its entirety. In question 1, the subject succeeded in changing the equation to $b - a - c = 1$. However, when testing the zero vector, he wrote 0 + 0 + 0 = subspace $R^3 = 1$. If you are in the Object Stage you will immediately realize that $0 \neq 1$ This should give rise to the conceptual conclusion that the element of identity is not contained in it, so the set is not subspace. Instead, the subject writes a floating sentence that is strong evidence that the subject does not understand the structural nature of the subspace itself. In question 2, the subject only recopies the information from the question, which is the general form (a, b, c, d) along with the two equations of obstacles. Let alone viewing the base as an Object, the subject doesn't even start the process of parameter substitution to form its free vector.

4. Understanding of Subject Concepts at the Schema Level

AY subjects do not meet the Schema Level criteria. In question 1 even though the subject managed to write the final conclusion *"0 = 1 Not a subspace"*, The way to reach this conclusion actually proves the absence of the Scheme Stage. In question 2 failure to reach the Schema Stage (even failing at the most basic stage) is very transparent in solving this basic question.

Here are the answers from 2 subjects that show the errors of all stages of action, process, object, and schema.

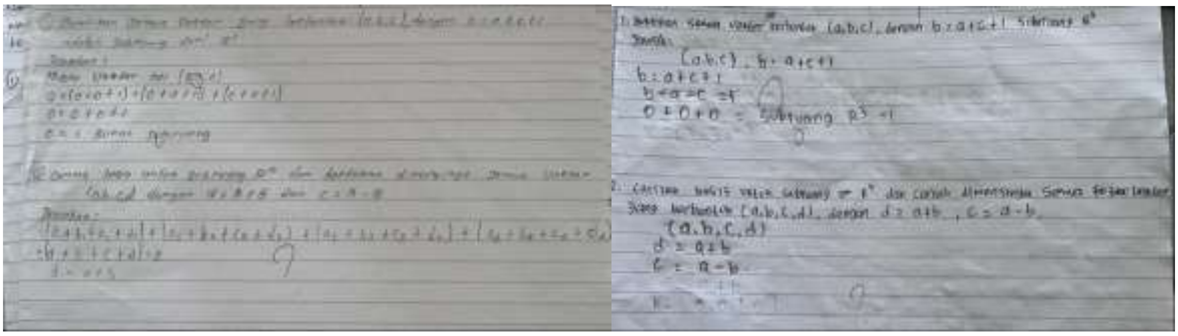

Figure 1. Answers to Understanding Algebraic Concepts for Low Ability Subjects

The medium-ability subjects

The results of the analysis of the answers of moderately capable subjects in 2 questions to see the students' understanding of the concept of Linear Algebra based on the APOS theory are described as follows:

5. Understanding Subject Concepts at the Action Stage

The NHP subjects in the Action Stage meet in question 1, but do not meet in question 2. In question 1, this subject performs substitution actions on target. It is the success of this Action that allows him to see mathematical contradictions $(0 \neq 1)$ and immediately draw the structural conclusion that the set is not subspace. Meanwhile, in question 2 the subjects actually wrote the pronunciation $i = (0,0)$ and $j = (0,0)$, then force the vector shape to be $v = a(0,0) + b(0,0) = ai + bj$. This is a form of dead memorization of the example of a space problem R^2 that he had seen.

Similarly, the RA subject in the Action Stage meets in question 1, but does not meet the Action Stage in question 2. In problem 1 because the substitution action is correct, this subject is able to see the mathematical contradiction $(0 \neq 1)$ clearly. Then respond to the result of the action by giving a cross (X) and immediately draw the conclusion that the set is not subspace. In question 2 to fulfill the Action Stage in this problem, the subject is required to substitute the created parameter into the remaining constraint equation, namely $d = a + b$ dan $c = a - b$. However, the subject simply rewrites the equation and fails to synthesize it so that it becomes $d = t + u$ and $c = t - u$.

6. Understanding of Subject Concepts at the Process Stage

NHP subjects met the Process Stage in Question 1, but failed completely in Question 2. It can be seen in question 1 that the strongest evidence that it is in the Process Stage (and starting to enter the Object) is the narrative of the explanation: *"in the calculation above the zero vector is not fulfilled because the result of the zero vector check is invalid* $0 + 0 + 1 = 1 \neq 0$*..."*. The subject is able to analyze the results of his own calculation and break the chain of proof without the need to check the other two conditions. In question 2 Stages of the Process the ability to synthesize the equation of constraints $(d = a + b \, dan \, c = a - b)$ into a common vector (a, b, c, d). However the subject does not perform this fusion/substitution at all to form a vector $(a, b, a - b, a + b)$.

Similarly, the RA subject in the Process Stage meets in question 1, but fails to completely meet the Process Stage in question 2. In question 1, the subject enters $(0,0,0)$ into the constraint function $b = a + c + 1$ appropriately be $0 = 0 + 0 + 1$. Ability to see results $0 \neq 1$, cross (X), and conclude *"not a Subspace of R^3 because it does not meet any of the conditions"* indicates that the mathematical procedure has been well internalized. In question 2, the subject should substitute the artificial variable into the equation *c* and *d*, thus forming a vector $(t, v, t - v, t + v)$. Since this crucial step of vector formation does not occur, the subject cannot process decomposition at all to eject *the variables t* and *v* to obtain the base vector.

7. Understanding of Subject Concepts at the Object Level

The NHP subject lacked the Object Stage in its entirety, because in question 1 it met the Object Stage, but in question 2 it failed to meet the Object Stage completely. In question 1, this subject successfully demonstrated thinking in the Object Stage. The subject registers the subspatial condition, choosing to test the identity element (vector zero). His ability to write *"in the above calculation the zero vector is not fulfilled... means it does not load zero. So obtained... not subspace"* indicates that the subject views this proof procedure as a single logical entity intact. The subject knows that the collapse of one foundation (zero vector) is enough to execute a conclusion without having to waste time calculating the conditions of closure. In question 2 instead of doing the correct algebraic process, the students made up a very chaotic manipulation of symbols by making up the $i = (0,0)$ dan $j = (0,0)$. Then force the vector shape to be $v = a(0,0) + b(0,0) = ai + bj$. At the end of the line, the subject concludes *"So* S *is the base... and its dimensions* = *2".* Although the number 2 for the dimension happens to be correct in the final result, this conclusion is not drawn from the evaluation of the properties of the Object (base vectors) found. This is just a guess.

Similarly, the RA subject also does not meet the Object Stage in its entirety, because in question 1 it meets the Object Stage, but in question 2 it completely fails to meet the Object Stage. In problem 1, because the subject is able to view the proof procedure as a single logical entity, the whole logic and the failure of one fundamental axiom is enough to break the chain of calculations is an indicator of meeting the Object Stage. In question 2 Someone at the Object Level will see a parametric vector $(a, b, a - b, a + b)$ as a unit that can be broken down, instead of guessing and randomly entering specific numbers into the general base search, by supposing the parameters $a = t$ dan $b = v$ then stop completely. This subject cannot perform the vector decomposition process so it fails to reach the object stage.

8. Understanding of Subject Concepts at the Schema Level

NHP subjects have not met the Scheme Stage criteria at all. In question 1 although on the whole answer sheet this subject managed to prove that the set is not a subspace through the contradiction of the zero vector, it is not the result of the thinking of the Schema Stage. Subjects with a mature Schema will instantly recognize the similarities $(0 \neq 1)$ $b = a + c + 1$ as a system of non-homogeneous linear equations. Without the need to calculate at length, his structural intuition would say that geometrically, it is a plane that shifts from the point of origin $(0,0,0)$, so that it is impossible to be subspace. Therefore, it purely relies on the memorization of the "zero vector check" procedure. In question 2, the subject forced the translation i and j into the vector to be $a(0,0) + b(0,0)$. This is clear evidence that his concepts of Linear Algebra are intertwined and unstructured in a correct Scheme.

Similarly, the RA subjects also did not meet the criteria of the Scheme Stage at all. In question 1, the subject succeeded in subordinating $(0,0,0)$, finds a logical contradiction 0 1,

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and immediately stops the proof procedure to draw the conclusion that the set is not subspace. This is an excellent Object reasoning, not a schema level. In question 2 the total failure of the Schema Stage looks very extreme, because the subject fails even to pass the most basic cognitive stage (Action Stage).

The following are the answers from 2 subjects that show that they still do not meet the stages of action, process, and objects, and do not meet the schema stage in the subjects of moderate ability.

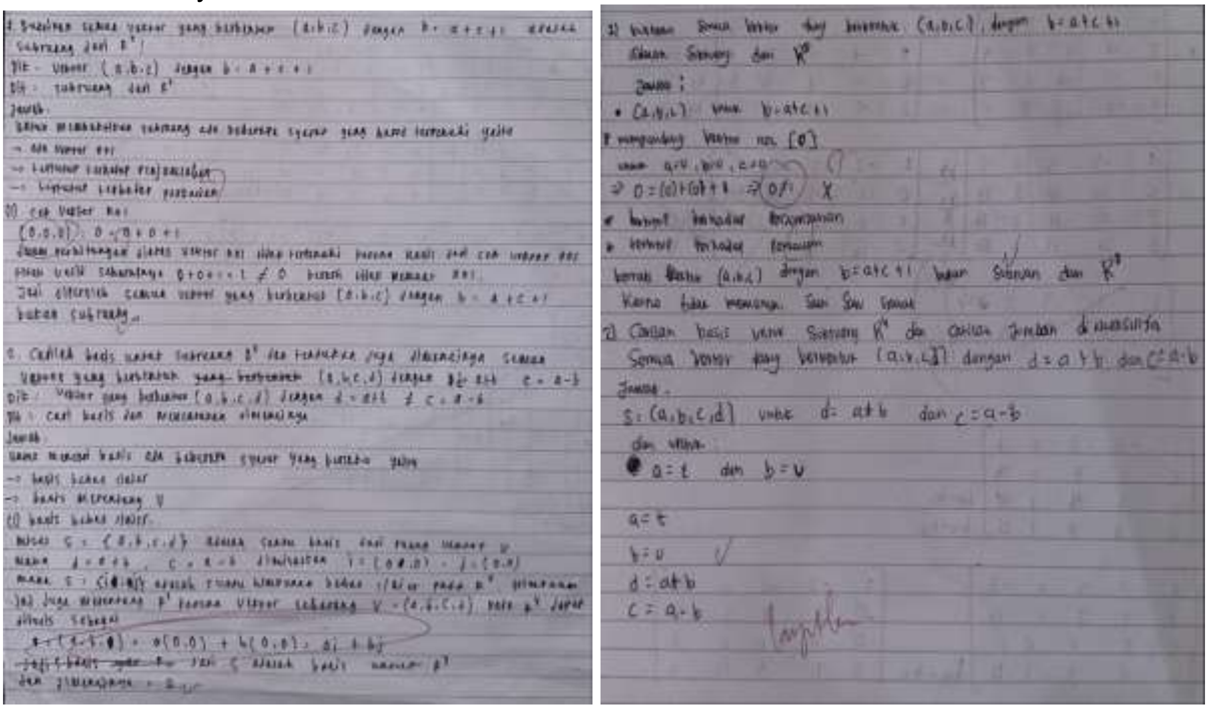

Figure 2. Answers to Understanding Algebraic Concepts Subject Medium Ability

The High-ability Subject

The results of the analysis of the answers of high-skilled subjects on 2 questions to see students' understanding of Linear Algebra concepts based on APOS theory are described as follows.

9. Understanding Subject Concepts at the Action Stage

NS subjects in the Action Stage meet both question 1 and question 2. In question 1, the subject performed intelligent procedural manipulation. The subject first subordinates the conditions $b = a + c + 1$ into the common vector form into $(a, a + c + 1, c)$. This is a very appropriate Action. In question 2, the subjects succeeded in performing an important initial action, which was to subordinate the constraint equations c and d into a general vector into a general vector $(a, b, a - b, a + b)$. It also performs parameter scaling. $a = r$ dan $b = s$ become $(r, s, r - s, r + s)$. Getting here, the action is valid.

The LR subject failed to meet the Action Stage in question 1, but in question 2 it was met. At the earliest stage 1 requires the subject to perform concrete mathematical actions, such as substituting a vector of zero $(0,0,0)$ into the equation $b = a + c + 1$. The subject does not perform this algebraic manipulation at all. In question 2, the subjects executed the algebraic manipulation very precisely and thoroughly. Subject defines vector (x_1, x_1, x_3, x_4), synthesize the given constraints, and decompose the column matrix based on its parameters $[a\ b\ \ a - b\ a + b] = [a\ 0\ \ a\ a] + [a\ b\ \ -b\ b] = a[1\ 0\ \ 1\ 1] + b[0\ 1\ \ -1\ 1]$.

10. Understanding of Subject Concepts at the Process Stage

NS subjects at the Process Stage meet in question 1 but fail in question 2. In question 1, zero vector testing is carried out by inserting $a = 0$ and $c = 0$, that produce a valuable middle

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component 1 $(0+1+0 = 1)$. The subject realizes that $1 \neq 0$, so that the set fails to contain an identity element. In question 2 when trying to extract (decompose) the parameters r and s to form a linear combination, students experienced a severe error. The subject writes the result of decomposition as $r(1,0,0) + s(0,1,0)$. The subject reduces the vector that should be in the R^4 become a standard base R^3 . This proves that the process of extracting the resistor vector is completely incomprehensible.

The LR subject in the Process Stage failed in question 1, but met in question 2. In question 1 because no action was taken, there was no procedure that could be evaluated. Reasons given, *"Because $b = a + c + 1$ on axioms 4 and 5... not equal to 0"*, indicates procedural errors. Subjects use the term "axiom" randomly without coherent step-by-step proof. In question 2, the subjects show a solid algorithmic process. The scribble at the beginning of the answer, where he had tried to use the standard basis i, j, k, l, was actually positive cognitive evidence. The snippet indicates *self-correction* and then autonomously redirects the Process to the correct parameter extraction flow.

11. Understanding of Subject Concepts at the Object Level

The NS subject at the Object Stage meets in question 1 but fails completely in question 2. In question 1, the subject's conclusion narrative is strong evidence of the achievement of the Object Stage. The subject wrote: *"because it is not proven... so that it cannot prove that the next step is closed to addition and multiplication"*. The subject views the subspace structure as a whole entity; The destruction of one basic axiom is enough to abort the entire system without the hassle of calculating the other axioms. In question 2, the subject closed his answer by writing ∴ basis = 2. This is a fatal conceptual error that proves the absence of an Object Stage. The subject confuses the definition of "Basis" (which is a set of vectors) with "Dimension" (which is a number/cardinality). The subject does not view a base as a set of vector objects that span space.

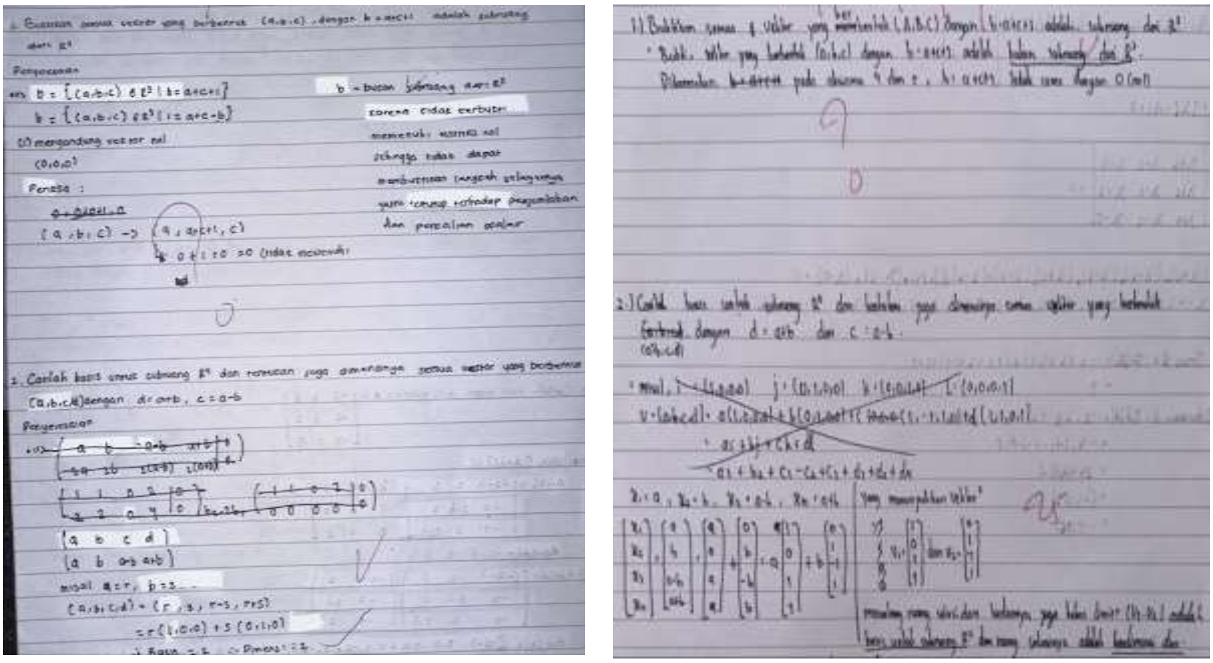
The LR subject at the Object Stage does not meet in question 1 but meets in question 2. In question 1 without an operational and procedural foundation, the understanding of subspace as a mathematical structure (object) is not formed in this answer. The big red question mark (?) streak from the examiner validates this very well. In question 2 the subjects are able to treat the final result of their algebraic manipulation as a new abstract entity. Subject Packaging Vector $[1\ 0\ 1\ 1]$ dan $[0\ 1 -1\ 1]$ become an object v_1 and v_2 then assigns properties to the object, namely "spanning the solution space" and "linear free".

12. Understanding of Subject Concepts at the Schema Level

NS subjects in the Schema Stage do not meet in question 1 and do not meet in question 2. In question 1, although the logic is very complicated, there is a slight gap in the scheme regarding terminology. The subject uses the term *"zero matrix"* when the context is a set of vectors in R^3 . However, conceptually the proof is valid. Question 2 of the matrix snippet at the beginning of the answer, where the subject tries to insert parametric equations into a random numerical matrix shows that the concepts of Linear Algebra (such as matrices, vectors, bases, and parameters) still collide with each other and have not yet formed a coherent Scheme of Solution.

The LR subject in the Schema Stage does not meet in question 1 but meets in question 2. In question 1 for the same reason when the object stage is without an operational and procedural foundation, the understanding of the subspace as a mathematical structure of the scheme is not formed in this answer. In question 2 at the end, the subject strings together all the acts of counting and understanding his concepts into a whole network of Schemas. The subject relates the results of the extraction of the parameters as a basis, and precisely calculates the cardinality to conclude that the dimensions are two. The conclusion sentence is structured, uses accurate Linear Algebra terminology, and logically answers the question instructions.

The following are the answers from 2 subjects that show an overview of the Action Stages, Processes, Objects, and Schemes in the subjects of high ability.



Gambar 3. Jawaban Pemahaman Konsep Aljabar Subjek Kemampuan Tinggi

Based on the results of the analysis that have been described for the subjects of low ability stages, the APOS theory both Actions, Processes, Objects, and Schemas are not fulfilled at all. These findings are in line with the research of Maulina et al. (2025) which confirms that low-ability subjects often experience cognitive paralysis from the Action Stage, namely the inability to perform substitution manipulation and basic algebraic computation (Maulina et al., 2025). Operational failures in this Action Stage are hierarchical and deterministic. As supported by the findings of Fitria (2024), the absence of a valid Action trigger causes no mechanism to occur, so that the group of low-skilled subjects is certain to fail to construct a complete understanding, both at the Process Stage, Object Stage, and Schema Stage (Fitria et al., 2024).

Subjects with a moderate level of ability have a fragmented/partial understanding of the concept of Linear Algebra. Subjects are able to reach the Object stage of evaluative problem solving (testing or proof), but experience failure in the Action stage when faced with constructive problem-solving (abstraction and the formation of new models). Since understanding is only a collection of procedures that are not interconnected, the two subjects do not meet the criteria of the Schema Stage. The findings regarding the understanding of terptah/prial in moderately capable subjects are in line with Supratman (Supratman et al., 2022) and Yunita (Yunita et al., 2024). Subjects in this category are generally stuck in the concept of memorized symbol manipulation, where they are able to perform the Process Stage and partial Object Stage on evaluative problems whose algorithms are routine. However, subjects encounter epistemological obstacles when faced with abstract construction problems such as the determination of parametric bases. The absence of connections between the concepts of Linear Algebra proves that the mental scheme as a whole is not formed in students with moderate ability.

Subjects with a high level of ability show very contrasting performance between understanding concepts. NS subjects are very proficient in the evaluative problem solving of axiomatic proof at the Object Stage, but experience fatal misconceptions in the problem solving of base construction, so that they fail in Process and Object. In contrast, LR subjects had a complete failure at proving the basic axioms, thus failing at the Action Stage, but were able to achieve perfect cognitive mastery until they reached the Schema Stage on the decomposition of the parametric matrix. The findings of conceptual understanding in high-ability subjects are in line with various findings in A. Astuti (Astuti & Zulhendri, 2017) and Supratman (Supratman et al., 2022), which state that even superior subjects often experience a fragmented / partial understanding. The high level of construction problem-solving does not automatically guarantee the maturity of axiomatic proofing evaluative problem-solving. Furthermore, the clash of concepts such as confusing the terminology of the base with dimensions or vectors with the matrix is an obstacle that confirms that the Schema Stage regarding axiomatics has not been fully grasped, although it is able to reach the Object Stage in a constructive concept.

CONCLUSION

The achievement of a student's thinking structure varies greatly depending on the level of ability. In general, most students experience a partial understanding and face obstacles in forming a complete Schema Stage framework, The following is the understanding of students' Linear Algebra concepts based on the student's ability categories is as follows:

Students with Low Ability experience cognitive failure from the beginning so that they do not meet the criteria of the Action Stage, Process, Object, or Scheme at all. These failures are hierarchical and deterministic, the inability to manipulate algebraic computations and basic substitutions in the Action Stage results in the absence of a construction mechanism for the later stages of understanding.

Students with Moderate Ability have a divided or partial understanding of concepts. Students are able to reach the Process Stage and partial Object Stage on evaluative problems whose algorithms are routine, but experience failure in the Action Stage when faced with abstract constructive problem solving. Because understanding is only a collection of memorized and uninterconnected procedures, the Schema Stage as a whole is not formed.

Students with High Abilities show mastery of opposite concepts. Students with high abilities reach the Schema Stage on a constructive problem, but fail completely at the Action Stage on a basic axiom proof problem, or vice versa. The high problem-solving ability of algebraic constructions does not automatically guarantee the evaluative maturity of axiomatic proofs. Confusing base terminology with dimensions is an obstacle that confirms that the Schema Stage has not been met.

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